VANIER COLLEGE
DEPARTMENT OF MATHEMATICS
DIFFERENTIAL EQUATIONS, H15, FINAL EXAMINATION

(1) (3 marks) Determine the general real solution of the system of DE's

$$\begin{aligned} y_1' &= 6y_1 - y_2 \\ y_2' &= 5y_1 + 4y_2 \end{aligned}$$

(2) (3 marks) Find the fundamental matrix $\Phi(t)$ satisfying $\Phi(0) = \mathbf{I}$ for the following system

$$\begin{aligned} y_1' &= -3y_1 + y_2 \\ y_2' &= 2y_1 - 4y_2 \end{aligned}$$

Use this fundamental matrix to solve this system of DE's with initial conditions $y_1(0) = 1, y_2(0) = -1$.

(3) (3 marks) Determine the general solution of the nonhomogeneous system of DE's

$$\begin{aligned} y_1' &= -3y_1 + y_2 + 3t \\ y_2' &= 2y_1 - 4y_2 + e^{-t} \end{aligned}$$

Notice that the coefficient system is the same as in Problem 2, so you can reuse the pertinent quantities computed there.

(4) (3 marks) Compute the relevant eigenvalues, and if the eigenvalues are real, also the eigenvectors and then sketch the phase portraits of the following systems of DE's

$$a)\ \mathbf{y}' = A\mathbf{y},\ \ A = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix} \qquad b)\ \mathbf{y}' = B\mathbf{y},\ \ B = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$$

(5) (3 marks) Solve the initial value problem

$$\begin{array}{rcl} y_1' & = & 3y_1 - 18y_2 \\ y_2' & = & 2y_1 - 9y_2 \end{array}, \qquad y_1(0) = -1,\ y_2(0) = -1.$$

(6) (3 marks) Consider the system of DE's depending on a parameter α

$$\begin{aligned} y_1' &= 5y_1 + 3y_2 \\ y_2' &= \alpha y_1 + 5y_2 \end{aligned}$$

a) Determine the critical values of α where the qualitative nature of the phase portrait for the system changes.
b) The critical values split the real axis into intervals. Draw a phase portait for the system for α in each of these intervals.

(7) (3.5 marks) Determine the first three terms of the two series solutions of the DE

$$y'' + 2\cos(x)y = 0$$

centered ar $x = 0$. Remember the Maclaurin series $\cos(x) = \sum_{n=0}^{\infty}(-1)^n x^{2n}/(2n)!$

(8) (3.5 marks) Consider the Legendre equation

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0.$$

a) Determine the recurrence relation for the coefficients of the power series solution of this equation centered at $x = 0$.
b) Argue that, when α is a nonnegative integer, the Legendre equation has a polynomial solution.
c) Determine the Legendre polynomial corresponding to $\alpha = 4$ up to a multiplicative constant.

(9) (2 marks) Consider the DE

$$y'' + \frac{4}{(t+1)(t^2+1)}y' + \frac{1}{t^2+1}y = 0.$$

If $y(t) = \sum_{n=0}^{\infty}(t-2)^n$ is a power series solution for this DE determine a lower bound for its radius of convergence.

(10) (3 marks) For the the DE

$$3ty'' + y' - y = 0$$

a) Find the indicial equation and the recurrence relation.
b) Find the the first three terms of the series solution, for $t > 0$, corresponding to the larger root of the indicial equation.