DISCRETE MATHEMATICS, H15, TEST 2

(1) (2.5 marks) A certain computer algorithm executes four times as many plus two operations when it runs with an input of size k as when it runs with input of size $k-1$ (where $k \in \mathbb{N}, k > 1$). When the algorithm is run on input of size 1, it executes 3 operations. Set up a recurrence relation for the number of operations on input of size k.

Using iteration derive a formula for the number of operations the algorithm runs when it is executed on an input of size n. Prove your formula by induction.

(2) (2 marks) Derive the following equality between sets algebraically using Boolean algebra identities. Give a reason for every step

$$(A \cup C) \cap [(A \cap B) \cup (B \backslash C)] = A \cap B.$$

(3) (2 marks) Prove or disprove the following statement: for any set A, $\mathcal{P}(A \times A) = \mathcal{P}(A) \times \mathcal{P}(A)$.

(4) (2.5 marks) Prove that the set $\mathbb{N}^2$ is countably infinite. Now argue by induction that the set $\mathbb{N}^n$ is countably infinite for every natural number n.

(5) (2 marks) How many poker hands contain either {three face cards and two aces} or {at least four cards of the same suit}?

(6) (2.5 marks) Given a set of 52 distinct integers, show that there must be two whose sum or difference is divisible by 100.

(7) (2.5 marks) Jana manages 12 programmers, 4 senior and 8 junior, at Sofitel Inc.
a) In how many way can Jana form a team of 5 programmers?
b) In how many ways can Jana select 5 programmers for 5 different tasks (no programmer can do more than one task).
c) In how many ways can Jana select a team of 5 programmers including at least two senior programmers?
d) Two of the senior programmers, Jake and Jane refuse to work on the same team. In how many ways can Jana select a team of 5 programmers including at least two senior programmers if Jake and Jane cannot be together on the team?

(8) (2 marks) Think of a set with $m+n$ elements as composed of two parts, one with m elements and the other with n elements. Give a combinatorial proof of the identity

$$_{m+n}C_r = \sum_{i=0}^{r} {}_mC_i \cdot {}_nC_{r-i},$$

where $m, n \in \mathbb{N}$ and r is a nonnegative integer less than or equal to both m and n.

(9) (2 marks) a) A florist has a large number of roses, carnations and lilies in stock. How many different bouquets of containing ten flowers can be made?
b) Find the coefficient of $x^4y^2z^5$ in the expression $(x/2 - 2y + z)^{11}$.

(10) (2.5 marks) Check that $5|(m^2 - n^2)$ is an equivalence relation on $\mathbb{Z}$. Describe the equivalence classes of this equivalence relation.

(11) (2 marks) Prove that the relation ρ on $\mathbb{N}$ defined as follows:

$$m\rho n \leftrightarrow m|n, \quad m, n \in \mathbb{N}$$

is a partial order on $\mathbb{N}$. Draw the Hasse diagram of this partial order when restricted to $\{1, 2, 3, 4, 5, 6, 7\}$.

(12) (2.5 marks) Your private RSA key is $p = 7, q = 13, s = 5$. Using your public key $n = 91, s = 5$ an alluring acquaintance has send you a message including the time of a proposed meet up, $E = 81$, which has been encoded with your public key. At what time are you invited to meet, decode E.