VANIER COLLEGE, DEPT. OF MATHEMATICS
LINEAR ALGEBRA, A15, FINAL EXAM

Name:______________________________ Student Number:____________________

(1) (3 marks) Determine the equation of the unique cubic polynomial whose graph passes though the points $(-2,0),(-1,4),(1,6)$ and $(2,40)$.

(2) (3 marks) Solve the following matrix equation for a 2×2 matrix X:

$$\frac{1}{2}AX^{-1} = (2X - A^T)^{-1}, \quad \text{where} \quad A = \begin{pmatrix} 2 & -4 \\ -1 & 3 \end{pmatrix}$$

(3) (4 marks) Consider the skew lines $L_1 = \{x = 1 + t, y = 2 + t, z = 3\}$ and $L_2 = \{x = 2 + s, y = 1 - 3s, z = -1 - 2s\}$.
a) Determine the shortest distance from L_1 to L_2 and the points A and B on the two lines which are closest to each other.

b) Determine the equation of the line through A and B.

c) Determine the equation of the plane containing A and L_2.

(4) (3 marks) a) Compute the inverse of the matrix

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{pmatrix}$$

using the formula with the adjugate.

b) Use A^{-1} to solve the matrix equation

$$XA = \begin{pmatrix} 2 & -1 & 0 \\ 4 & 0 & -1 \end{pmatrix}$$

(5) (3 marks) Consider the paralellogram $Para$ generated by the origin and the points $A = (-2, 3)^T$ and $B = (2, -5)^T$.
a) Determine the area of this parallelogram.

b) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation that maps the point A to the point $C = (4, 6)^T$ and also maps the point B to the point $D = (-8, -12)^T$ determine the matrix M_T of this linear transformation.

c) Determine the area of the image of the $Para$ under T and confirm that the area has been rescaled by the determinant of M_T.

(6) (3 marks) i) Let $A \in M_{3\times 3}$ be a fixed matrix. Is the set of matrices $\mathcal{U} = \{X \in M_{3\times 3} | XA - AX = 0\}$ is a subspace of $M_{3\times 3}$? Show the details of your argument.

ii) Is the set of all polynomials $p(x)$ in P_3 such that $p(-x) = p(x)$ a subspace of P_3? Show the details of your argument.

(7) (4 marks) For the matrix

$$A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 2 & -1 & 1 & -2 \\ 3 & -4 & 4 & -5 \end{bmatrix}$$

give an orthogonal bases for imA and for $kerA$.

(8) (4 marks) Consider

$$\mathcal{U} = \left\{ X_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \ X_2 = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}, \ X_3 = \begin{pmatrix} 12 \\ 5 \\ 0 \end{pmatrix}. \right\}$$

a) Determine an orthonormal basis $\mathcal{B}$ for $\mathcal{U}$.

b) Describe $\mathcal{U}$ in geometrical terms.

c) Determine the coefficients of the following two vectors in the basis $\mathcal{B}$ you constructed:

$$Y = \begin{pmatrix} 8 \\ 3 \\ 2 \end{pmatrix}, \ Z = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}.$$

(9) (3 marks) Is $\mathcal{B} = \{1 + x, 1 - x + x^2, 3 + x^2\}$ is a basis of $\mathbf{P}_2$. If yes, expand $p(x) = -1 + 3x + x^2$ in this basis.

(10) (2 marks) Let A be an $m \times n$ matrix. For each of the following cases answer the following four questions by filling in the table below.

size A	3×5	5×3	4×4	3×3	5×3
rank A	3	3	3	3	2
Q1					
Q2					
Q3					
Q4					

Q1) Is the linear system $AX = B$ consistent for every possible choice of B?
Q2) If the linear system is consistent for some specific B, how many parameters are there in the general solution?
Q3) Is it true that $imA = \mathbb{R}^m$?
Q4) Is it true that $kerA = \{0\}$?

(11) (2 marks) Let A be $n \times n$ matrix. Explain what could be said about the values of n and $det\ A$ if $A^2 + 2I = 0$.

(12) (3 marks) a) Show that for all vectors $X, Y \in \mathbb{R}^n$ the following identity holds

$$X \cdot Y = \frac{1}{4}\|X + Y\|^2 - \frac{1}{4}\|X - Y\|^2$$

b) Prove the following statement: If the diagonals of a parallelogram have equal length, then the parallelogram must be a rectangle.